Mapping the Spacetime Metric with a Global Navigation Satellite System (Study Reference Number: 09-1301)

Type of activity: Standard study (25 k€)

Background and motivation

Introduction

Global Navigation Satellites System (GNSS) is a generic term for satellites navigation systems. The Global Positioning System (GPS) is currently the only fully operational GNSS, developed and operated by the United States department of defense. GLONASS is the Russian GNSS, which is in the process of being restored, with the help of India. Galileo, the European Union GNSS, is currently in the phase of deployment, and scheduled to be operational in 2013. A new generation of the US GPS, GPS III, is currently under development.

The basic principle of these positioning systems is based on a Newtonian conception of time and space. The signals from four satellites are needed for a receiver to determine accurately its position. When it receives the signal from one satellite, the receiver can determine the time elapsed between the emission and the reception of the signal with a clock, and via the time difference calculate its distance R with respect to the satellite. With the knowledge of the position of the satellite at the moment of the emission, the receiver knows its position is on the surface of a sphere centered on the satellite and of radius R. With four signals (four spheres) its exact location can be determined [1].

The need of relativistic mechanics for GNSS

The classical concept for GNSS would work ideally if all satellites and the receiver were at rest in an inertial reference frame. But at the level of precision needed by a GNSS, one has to consider inertial effects and curvature effects of spacetime, which are far from being negligible [1]. It is well-known that in this case, space and time cannot be considered as absolute. Then there are at least two very different ways of defining a positioning system.

One way is to try to preserve the Newtonian conception of absolute time and space. One keeps the method described in the introduction, but adds corrections coming from general relativity, at the level of accuracy desired. The clocks of the satellites and of the ground stations tracking the satellites have to be synchronized in an underlying local inertial frame. To compute the corrections, one needs to model an ideal Earth and to expand the solution of the Einstein equation at the order of accuracy desired. This leads to numerous corrections, depending on the kinematics of the positioning system. The most important ones are the first and second order Doppler frequency shifts, the gravitational frequency shifts and the Sagnac effect. Several additional effects have to be considered at the level of a few centimeters precision, which is the ultimate precision of the GPS. For a detailed description of these corrections see [1].

The natural evolution of GNSS is to become more and more accurate with the help of very stable clocks, and autonomous (no need for ground stations) with the help of cross links between the satellites (as for GPS III). The drift of a GPS Cesium clock is about 4 ns after one day. Galileo will embark hydrogen maser clocks, with a drift of about 1 ns after one day. But state of the art atomic clocks are far more stable than these: e.g. the Cesium clock Pharao, which will be embarked on the International Space Station (ISS) will have a drift of 26 ps/day. Optical clocks are the most stable clocks today, with a drift of only 0.3 ps/day, although their accuracy is not yet better than the one of Cesium clocks. At this low level of uncertainty a lot of supplementary corrections have to be added, mainly the Shapiro time delay, and the effects of the quadrupole moment and the angular momentum of the Earth [2].

The relativistic positioning system

Another way to define the positioning system is to abandon the Newtonian concept of absolute space and time, which is known to be a classical approximation, and to define a relativistic positioning system, with the so-called emission or GPS coordinates [3-10]. A project called SYPOR, "Système de Positionnement Relativiste", has been proposed by B. Coll and collaborators [11]. The relativistic positioning system is composed of four satellites broadcasting their proper times by means of electromagnetic signals. The coordinates of the receiver are simply the four proper times $\{\tau_1,\tau_2,\tau_3,\tau_4\}$ received from the four satellites. Then the receiver knows his trajectory in these emission coordinates. Moreover, if each satellite broadcasts its proper time to the other satellites, the receiver knows also the trajectories of the satellites, and the system is autonomous.

The emission coordinates have very attractive properties. First of all, these are covariant quantities; they are independent of the observer, although dependent on the set of satellites one chooses and on their trajectories. The set of satellites constitutes a primary reference system, with no need to define a terrestrial reference frame; so there is no need to track the satellites with ground stations or to synchronize the clocks. There is no need also for relativistic corrections, as relativity is already included in the definition of the positioning system. If needed, they can be related to more usual terrestrial coordinate system.

More than a positioning system

The next generation of GNSS will have cross-link capabilities. Each satellite, as well as broadcasting their proper time, will be able to broadcast the proper time of each other satellite in view. Then, any of the satellites knows its own trajectory as well as the trajectories of the other satellites. With this information, one can infer what is the spacetime metric acting on the constellation of satellites [12]. The information on the spacetime metric can come from different sources: four clocks are needed to define a primary reference system; the other ones are used to monitor the spacetime metric. Additional information on the spacetime metric can be obtained if the satellites have an accelerometer, a gyrometer or a gradiometer. The problem of recovering the spacetime metric is an inverse problem: one wants to recover the spacetime metric from the observed data. A. Tarantola *et al.* give a possible algorithm to solve this inverse problem.

Research and Study Objectives

The basic unknowns of the problem are the components of the contravariant metric in the emission coordinate system. The metric must satisfy several constraints: its diagonal components in the emission coordinates system must be null, it must satisfy the Einstein equations and it has to be smooth. The primary constellation, consisting of four satellites, defines the emission coordinates system. The secondary constellation is constituted by the satellites that do not contribute to define the emission coordinates. The data of the problem are mainly the proper times of the satellites broadcasted to the other satellites. These signals are electromagnetic waves, so their trajectory must be a null geodesic. Knowing the trajectory of a satellite, and the proper time of emission of another satellite, one can solve the geodesic equation, derive a *computed* proper time of arrival, and compare it with the observed one. Then an algorithm can be defined in order to optimize our knowledge of the spacetime metric. In case the satellites are not in free-fall, additional data coming from an accelerometer or a gyrometer could be used, giving additional information on the connection coefficients; moreover a gradiometer could be used, giving additional information on the Riemann tensor. **The aim of this study is to develop such an algorithm**, and to characterize the influence of the clock noise on the inferred metric. To achieve this goal, the following three main steps are proposed:

1. In order to simulate the data, the spacetime metric components have to be computed in the emission coordinates:
    - We can assume that the components of the metric are known in an initial chosen coordinate system. This metric can be either an analytical or a numerical solution of the Einstein equation. The stress-energy tensor can model either vacuum or a perfect fluid, which would be the gas of the upper atmosphere.
    - Calculate the trajectories of the satellites in the principal constellation (parameterized by their proper time). For this one has to choose the initial conditions of the primary constellation. It is possible to study the influence of the chosen initial conditions, or of non-zero accelerations, coming from non-gravitational forces.
    - Solve the time transfer problem in order to calculate the coordinate transformations between the initial coordinate system and the emission coordinates.
2. Simulate the data
    - Compute the trajectories of the primary and the secondary constellation in the emission coordinate system, parameterized by their proper times. The initial conditions of the secondary constellation will have an influence on the spacetime domain where the metric will be inferred. One should try to optimize the coverage. Again the influence of non-gravitational forces can be studied.
    - Compute the arrival time data, by linking the proper times of each pair of trajectories by solving the time transfer problem.
    - Add some realistic clock noise (phase noise and drift) to the clocks of the secondary constellation, and possibly of the primary constellation. Possibly add acceleration and rotation noise, so that the satellites are not Fermi-Walker transported.
3. Recover the spacetime metric
    - Implement an algorithm, based on the one proposed by Tarantola et al. [12], with the following suggested improvements: addition of some constraint on the metric components, use a Kalman filter. Applicants are free and encouraged to propose and scientifically argue for different improvements than these in their proposal.
    - Assess possible uses of the recovered spacetime metric. In particular, one can infer the gravitational potential by applying a coordinate transformation to a Fermi frame [8].
    - Assess the influence of the clock noise and drift on the products.

All these steps can be done numerically. However, the procedure will have to be checked with an analytical resolution of the problem, which can only be done for a Minkowski metric. However, preliminary work within the ACT has shown that for some particular configuration of the constellation, a semi-analytical resolution of the problem can be done for the Schwarzchild metric, which will allow checking the numerical procedure up to certain accuracy. Since the simulation and the analysis of the data have similar procedures, they should be carried out in completely different ways (different algorithms, programming language...) in order to show the robustness of the data analysis.

Collaboration with the Advanced Concepts Team

This study is mainly addressed to researchers in the field of general relativity with background in numerical relativity and inverse problems. The project will be conducted in tight scientific collaboration with ACT-researchers. Next to the scientific discussions, ACT-researchers will develop a code to solve the problem for simple configurations of the satellite constellation, to check the robustness of the developed algorithm.

References

1. Neil Ashby, *Relativity in the Global Positioning System*, **Living Rev. Relativity** 6, 2003, http://www.livingreviews.org/lrr-2003-1, (cited on 03/2009)
2. Linet, B. and Teyssandier, P., *Time transfer and frequency shift to the order $1/c^4$ in the field of an axisymmetric rotating body*, **Phys. Rev. D**, 2002, Vol. 66
3. Rovelli, C., *GPS observables in general relativity*, **Phys. Rev. D**, 2002, Vol. 65(4)
4. Blagojević, M., Garecki, J., Hehl, F.W. & Obukhov, Y.N., *Real null coframes in general relativity and GPS type coordinates*, **Phys. Rev. D**, 2002, Vol. 65(4)
5. Coll, B. & Pozo, J.M., *Relativistic positioning systems: the emission coordinates*, **Class. Quantum Grav.**, 2006, Vol. 23
6. Alba, D. and Lusanna, L., *Generalized Radar 4-COORDINATES and Equal-Time Cauchy Surfaces for Arbitrary Accelerated Observers*, **Int. J. Mod. Phys. D**, 2007, Vol. 16
7. Pascual-Sánchez, J., *Introducing relativity in global navigation satellite systems*, **Ann. Phys. (Leipzig)**, 2007, Vol. 16
8. Bini, D., Geralico, A., Ruggiero, M.L. & Tartaglia, A., *Emission versus Fermi coordinates: applications to relativistic positioning systems*, **Class. Quantum Grav**., 2008, Vol. 25(20)
9. Ruggiero, M.L. & Tartaglia, A., *Mapping Cartesian Coordinates Into Emission Coordinates: Some Toy Models*, **Int. J. Mod. Phys. D**, 2008, Vol. 17
10. Lachièze-Rey, M., *The covariance of GPS coordinates and frames,* **Class. Quantum Grav.**, 2006, Vol. 23
11. Coll, B., *A principal positioning system for the Earth*, Capitaine, N. & Stavinschi, M. (ed.), Journées 2002 - systèmes de référence spatio-temporels. Astrometry from ground and from space, Bucharest, 25 - 28 September 2002
12. Tarantola, A., Klimes L., Pozo, J.M., and Coll, B., *Gravimetry, Relativity and the Global Navigation Satellite Systems lecture*, Int. School on Relativistic Coordinates, Reference and Positioning Systems held at Salamanca (Spain), 2005 (http://www.ipgp.jussieu.fr/~tarantola/Files/Professional/Teaching/Seminar/Texts/GNSS.pdf)